ON TWO CLASSES OF TRIGONOMETRICAL SERIES

BY

ARNE BROMAN

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ALMQVIST & WIKSELLS BOKTRYCKERI AB

ON TWO CLASSES OF TRIGONOMETRICAL SERIES

Inaugural Dissertation

by

ARNE BROMAN
FIL. LIC., ÖSTG.

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In publishing this dissertation I take the opportunity to express my gratitude to my principal teachers in mathematics at Uppsala university, Professors ARNE BEURLING and TRYGVE NAGELL. I then turn especially to Professor BEURLING, who has suggested this investigation and has followed my work with interest and encouragement, good advice and stimulating criticism.

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Uppsala in May 1947.

Arne Broman.

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Introduction

This paper is devoted to a study of the *trigonometrical series* (with real or complex coefficients)

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

which belong to either of two *classes* of trigonometrical series. We shall call these classes (S_α) and (T_α), where α is a parameter, $0 < \alpha < 1$. We define the classes by imposing the following conditions on the coefficients of the series:

(S_α) $$\sum_{n=1}^{\infty} n^\alpha (|a_n|^2 + |b_n|^2) < \infty$$

and

(T_α) $$\sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{n^\alpha} < \infty.$$

Any trigonometrical series belonging to a class (S_α) is evidently at the same time the *Fourier series* of a function $f(\theta)$, summable in Lebesgue's sense, such that for $n = 0, 1, \ldots$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

We shall also say that the function $f(\theta)$ belongs to the class (S_α).

On the contrary it is in general not possible to associate a summable function with a series of a class (T_α). The coefficients of the series are $o(n^{\alpha/2})$ but they do not necessarily satisfy any stronger condition. Thus the series can be divergent in every point.

In particular, the paper will be devoted to the extension to the case $0 < \alpha < 1$ of two theorems, proved by BEURLING in the case $\alpha = 1$. In this last case we shall denote the classes by (S_1) and (T_1). We shall refer to the two theorems as (A) and (B). The first theorem was published in 1939 ([1], page 5)[1]:

(A) *If the function $f(\theta)$ belongs to the class (S_1), then everywhere except perhaps on a set E of outer logarithmic capacity zero the Fourier series of $f(\theta)$ converges and its primitive function has a symmetrical derivative equal to the sum of the series.*

BEURLING has also remarked ([1], pages 9—10), that an extension of parts of theorem (A) to the case $0 < \alpha < 1$ is possible.[2] And quite recently (in 1946) SALEM and ZYGMUND have sharpened and extended this remark; they have in fact proved the following theorem ([4], page 27):

If the function $f(\theta)$ belongs to the class (S_α), then its Fourier series converges everywhere except perhaps on a set E whose capacity of order $1 - \alpha$ is zero.[3]

This question shall here be treated once more, as SALEM-ZYGMUND have not found the complete extension of the original BEURLING theorem. We shall in chapter 3 prove the following result by means of a representation formula, used by SALEM-ZYGMUND, and by simplifying their method (and it ought to be noted that the *symmetrical derivative* of theorem (A) can be replaced by the *derivative*):

(A′) *If the function $f(\theta)$ belongs to the class (S_α), then everywhere except perhaps in a set E whose outer capacity of order $1 - \alpha$ is zero the Fourier series of $f(\theta)$ converges and its primitive function has a determined derivative equal to the sum of the series.*[3]

The second BEURLING theorem was presented in a lecture at Uppsala university in 1940[4]:

[1] Numbers within brackets refer to the bibliography at the end of the paper.

[2] BEURLING, however, never persued the idea himself, but gave it to me as the subject for my dissertation.

[3] The definitions of the concepts of *capacity* and *outer capacity* can not conveniently be given and discussed in the introduction; the reader is referred to chapter 1.

[4] Neither theorem (B) nor its extension (B') were announced in the original BEURLING paper. As far as I know, they are the first theorems of their kind giving both necessary and sufficient conditions, disregarding trivial cases.

(B) *A necessary and sufficient condition for a closed set E to be a unicity set with respect to the class (T_1) is that the logarithmic capacity of E be zero.*

In theorem (*B*) unicity sets refer to ABEL-POISSON's summability method, not to convergence (the series belonging to the class (T_1) being in general divergent everywhere).

In chapter 4 we shall prove:

(B′) *A necessary and sufficient condition for a closed set E to be a unicity set with respect to the class (T_α) is that its capacity of order $1-\alpha$ be zero.*

The following theorem has also been proved by BEURLING ([1], page 6):

(C) *Let the function $f(\theta)$ belong to the class (S_1) and associate (e. g. by means of* POISSON's *integral) with $f(\theta)$ a function $f(r,\theta)$, harmonic in the unit circle. Then its total variation on the radius θ*

$$F(\theta)=\int_0^1\left|\frac{\partial f(r,\theta)}{\partial r}\right|dr$$

is a function of θ which also belongs to the class (S_1). Further $F(\theta)$ is finite everywhere except perhaps on a set whose outer logarithmic capacity is zero.

After having settled a preliminary question in chapter 1 we shall prove this extension of theorem (*C*) in chapter 2:

(C′) *Let the function $f(\theta)$ belong to the class (S_α) and define the function $F(\theta)$ as above. Then $F(\theta)$ belongs also to the class (S_α), and $F(\theta)$ is finite everywhere except perhaps on a set whose outer capacity of order $1-\alpha$ is zero.*

The original problem, given by BEURLING, concerned exceptional or singular sets rather than properties of trigonometrical series (compare [1], pages 1—2). Hence it is an interesting circumstance that the class (S_α) can be defined without reference to Fourier coefficients.

Suppose that $f(\theta)$ is a real or complex function belonging to the class $L^2(a, b)$; i. e. $f(\theta)$ is measurable and $f^2(\theta)$ is LEBESGUE integrable over an interval (a, b). And let $\omega_f(h)$ denote the following integral modulus of continuity:

$$\omega_f(h) = \int_a^{b-h} |f(\theta + h) - f(\theta)|^2 \, d\theta. \qquad (0<h<b-a)$$

It is a continuous function of h. Suppose further that $\omega_f(h)$ tends so rapidly to zero as $h \to +0$ that the integral

$$\int_0^{\delta} \frac{\omega_f(h)}{h^{1+\alpha}} \, dh \qquad (\delta>0,\ 0<\alpha<1)$$

is convergent. We shall then say that the function $f(\theta)$ belongs to the class (S_α).

In chapter 1 we shall see that this definition of the class (S_α) is equivalent to the former one (irrespective of the periodicity).

In the proofs of our theorems we shall again and again need some properties of a certain elementary function. In order that the derivation of these properties shall not disturb the main thread of the paper, we have collected them in a special final chapter which will be cited when needed. It is perhaps convenient for the reader to begin with that chapter.

It ought to be observed that we imagine all capacities to be measured on sets, situated on the circumference of the unit circle.

CHAPTER 1

Definitions. On the Large Values of Certain Summable Functions

1. — Consider a real function $f(\theta)$[1] of period 2π, belonging to L^2, and its Fourier series

$$f(\theta) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{1.1}$$

We shall say that *the function* $f(\theta)$ *belongs to the class* (S_α), $0 < \alpha < 1$, if the integral

$$S_\alpha(f) \equiv \int_0^{\infty} \int_0^{2\pi} \frac{[f(\theta + t) - f(\theta - t)]^2}{t^{1+\alpha}} \, d\theta \, dt < \infty. \tag{1.2}$$

Parseval's theorem gives

$$\int_0^{2\pi} [f(\theta + t) - f(\theta - t)]^2 \, d\theta = 4\pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \sin^2 nt,$$

and hence by integration with respect to t

$$S_\alpha(f) = \int_0^{\infty} \frac{dt}{t^{1+\alpha}} \cdot 4\pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \sin^2 nt =$$

$$= \sum_{n=1}^{\infty} n^\alpha (a_n^2 + b_n^2) \frac{4\pi n}{n^{1+\alpha}} \int_0^{\infty} \frac{\sin^2 nt}{t^{1+\alpha}} \, dt =$$

$$= \sum_{n=1}^{\infty} n^\alpha (a_n^2 + b_n^2) \cdot 4\pi \int_0^{\infty} \frac{\sin^2 t}{t^{1+\alpha}} \, dt,$$

[1] We exclude once for all the trivial case where $f(\theta)$ is a constant.

the last integral being convergent. Thus we can write[1]

$$S_\alpha(f) \equiv M \cdot \sum_{n=1}^{\infty} n^\alpha (a_n^2 + b_n^2) < \infty, \tag{1.3}$$

M depending only on α, and, when needed, we can use formula (1.3) as well as (1.2) to characterize the class (S_α).

Further consider the function associated with (1.1)

$$f(r,\theta) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta), \tag{1.4}$$

which is harmonic in the unit circle $(r < 1,\ 0 \leq \theta < 2\pi)$. We shall also say that $f(r,\theta)$ *belongs to the class* (S_α), if the series (1.3) is convergent.

In this chapter we shall study the set of points on the circumference of the unit circle where the function $f(\theta) \geq \varrho > 0$ when $f(\theta)$ belongs to the class (S_α).

2. — Suppose that E is an open or closed set on the circumference of the unit circle and that μ is a distribution of the unit mass on E (thus, μ is an additive non-negative set function, defined on $|z| = 1$, where $z = re^{i\theta}$, and further $\mu(E) = 1$ and $\mu(E^*) = 0$, where E^* is the set complementary to E). Suppose further that

$$u_\beta(r,\theta) = \int_0^{2\pi} \frac{1}{|re^{i\theta} - e^{i\varphi}|^\beta}\, d\mu(\varphi) \qquad (\beta = 1 - \alpha) \tag{1.5}$$

is *the generalized potential of order* β with respect to μ (compare Frostman, [2], § 10; compare also formula (5.1)). Where there can be no misunderstanding, we shall often denominate it briefly as *the potential* and denote it by u only. Let V_μ be the least upper bound of u, when the point (r,θ) is situated in the unit circle, and let $V(E)$ be the greatest lower bound of V_μ when μ varies:

$$V(E) = \inf_{\mu} \sup_{(r,\theta)} u_\beta(r,\theta). \tag{1.6}$$

[1] Concerning the use of the letters M and m in this paper see page 43.

By the *capacity of order* β of the set E we shall understand the quantity

$$C_\beta(E) = \frac{1}{V(E)^{1/\beta}}. \tag{1.7}$$

If E is an arbitrary set, we shall understand by its *outer capacity of order* β the greatest lower bound of the capacities of all open sets O containing E:

$$\bar{C}_\beta(E) = \inf_{O \supset E} C_\beta(O). \tag{1.8}$$

(Where there can be no danger of ambiguity we shall sometimes use the shorter terms *capacity* and *outer capacity*.)

3. — The above definitions of *capacity* and *outer capacity* are natural generalizations of definitions of the capacity, earlier given by DE LA VALLÉE POUSSIN ([5], page 226) in the case $\beta = 1$, and FROSTMAN ([2], page 49) in the case $0 < \beta < 1$.

The concept of *outer capacity* (which is of value when we want to show that a set is small) has been used earlier, to my knowledge, only by BEURLING (in 1939), MONNA (in 1939), LELONG (in 1942), and CARTAN (in 1942). MONNA has shown that $\bar{C}_\beta(F) = C_\beta(F)$ for a closed set F ([3], page 84).

FROSTMAN has shown ([2], § 34) that when the set E is *closed*, then the greatest lower bound $V(E)$ is attained by V_μ for a unique distribution μ. In this case, furthermore, $u = V(E)$ everywhere on E except perhaps on a subset of capacity zero. This μ is called the *equilibrium distribution* and u the *equilibrium potential*.

When the set E is *open*, then the greatest lower bound is in general not attained. Yet the capacity serves also in this case as a »measure» of E.

A set of capacity zero is always of LEBESGUE measure zero, but the converse does not hold: a set of measure zero can very well be of positive capacity.

4. — We now assert that the following theorem holds.

Theorem 1. 1° *Suppose that the real function* $f(\theta)$ *belongs to the class* (S_α) *and is normalized by the conditions* $S_\alpha(f) = 1$ *and* $a_0 = 0$. *Suppose further that* $f(\theta) \geq \varrho > 0$ *on an open set* E. *Then the*

following relation holds for its capacity of order $1-\alpha$ *(regarded on the unit circle):*

$$C_{1-\alpha}(E) \leq M \cdot \varrho^{-\frac{2}{1-\alpha}},$$

where M *depends only on* α. 2° *If the set* E *is arbitrary and* $f(\theta)$ *is lower semi-continuous, then the outer capacity of* E *satisfies the condition*

$$\bar{C}_{1-\alpha}(E) \leq M \cdot \varrho^{-\frac{2}{1-\alpha}}.$$

It ought to be noted that our condition that $f(\theta)$ is lower semicontinuous (or some other condition to the same effect) must enter into the formulation of theorem 1.2°. For if $f(\theta)$ belongs to the class (S_α) we can give to $f(\theta)$ the value $+\infty$ on a set of measure zero and of positive capacity and the value of the integral (1.2) then remains unchanged.

To prove theorem 1 we need a lemma.

5. — Lemma 1. *Let* μ *and* V_μ *have the above meanings. Let further*

$$\left.\begin{matrix} h_n \\ k_n \end{matrix}\right\} = \int_0^{2\pi} \left\{\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\} d\mu(\theta). \tag{1.9}$$

We then assert that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}(h_n^2 + k_n^2) \leq M \cdot V_\mu, \qquad (\alpha = 1-\beta) \tag{1.10}$$

where M *(has a new value and) depends only on* α.

Proof. By the definition of V_μ we have

$$V_\mu \geq \int_0^{2\pi} u(r,\theta)\, d\mu(\theta). \tag{1.11}$$

Formula (1.5) and the lemmas 7 and 6, pages 44 and 43, give

$$u(r,\theta) = \int_0^{2\pi} \frac{1}{|1 - re^{i(\theta-\varphi)}|^\beta} d\mu(\varphi) \geq \int_0^{2\pi} h_\beta(r, \theta-\varphi)\, d\mu(\varphi) >$$

$$> \int_0^{2\pi} \left\{\sum_{n=1}^{\infty} A_n r^n \cos n(\theta-\varphi)\right\} d\mu(\varphi).$$

A Stieltjes integral of a uniformly convergent series can of course be integrated term by term; hence for $r < 1$:

$$u(r, \theta) > \sum_{n=1}^{\infty} A_n r^n (h_n \cos n\theta + k_n \sin n\theta)$$

and for the same reason ($|h_n| \leq 1$ and $|k_n| \leq 1$ by (1.9)):

$$\int_0^{2\pi} u(r, \theta)\, d\mu(\theta) > \sum_{n=1}^{\infty} A_n r^n (h_n^2 + k_n^2).$$

Lemma 6, page 43, and formula (1.11) then give

$$V_\mu > m \sum_{n=1}^{\infty} \frac{1}{n^\alpha} r^n (h_n^2 + k_n^2),$$

or, this last series having only positive terms,

$$V_\mu > m \sum_{n=1}^{\infty} \frac{1}{n^\alpha} (h_n^2 + k_n^2),$$

which implies our assertion.

6. — *Proof of theorem 1.* 1° Given an arbitrary $\varepsilon > 0$. Let E consist of the disjoint open arcs $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$. Let β_k be a closed subarc of α_k. And let $f(r, \theta)$ have the same meaning as in (1.4). From a well known property of Poisson's integral we can choose $r_0 < 1$ so large that $f(r, \theta) > \varrho - \varepsilon$ on β_k when $r_0 < r < 1$. And so

$$\int_{\beta_k} f(r, \theta)\, d\mu(\theta) > (\varrho - \varepsilon)\, \mu(\beta_k) \qquad (r_0 < r < 1)$$

or

$$\liminf_{r=1-0} \int_{\beta_k} f(r, \theta)\, d\mu(\theta) \geq \varrho \cdot \mu(\beta_k),$$

from which

$$\liminf_{r=1-0} \int_{\alpha_k} f(r, \theta)\, d\mu(\theta) \geq \varrho \cdot \mu(\alpha_k),$$

for no mass is situated at the end points of α_k. Consequently

$$\liminf_{r=1-0} \int_0^{2\pi} f(r,\theta)\, d\mu(\theta) \geq \varrho \cdot \Sigma \mu(\alpha_k) = \varrho,$$

or for arbitrary $\varepsilon > 0$,

$$\varrho - \varepsilon < \int_0^{2\pi} f(r,\theta)\, d\mu(\theta), \tag{1.12}$$

if we give to r a value sufficiently near to 1.

Further we have (noticing that $a_0 = 0$)

$$\int_0^{2\pi} f(r,\theta)\, d\mu(\theta) = \int_0^{2\pi} \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)\, d\mu(\theta) = \tag{1.13}$$

$$= \sum_{n=1}^{\infty} r^n (a_n h_n + b_n k_n),$$

where we have once more integrated a uniformly convergent series term by term.

Schwarz's inequality and the condition $S_\alpha(f) = 1$ give

$$\left[\sum_{n=1}^{\infty} r^n (a_n h_n + b_n k_n)\right]^2 \leq \sum_{n=1}^{\infty} n^\alpha r^{2n} (a_n^2 + b_n^2) \cdot \tag{1.14}$$

$$\cdot \sum_{n=1}^{\infty} \frac{1}{n^\alpha} (h_n^2 + k_n^2) < \frac{1}{M} \cdot \sum_{n=1}^{\infty} \frac{1}{n^\alpha} (h_n^2 + k_n^2),$$

where M has the same value as in (1.3).

From the formulae (1.12), (1.13), (1.14), and from lemma 1 it follows that

$$(\varrho - \varepsilon)^2 < M \cdot V_\mu,$$

where M has a new value. But $\varepsilon > 0$ can be chosen arbitrarily small. Thus

$$\varrho^2 \leq M \cdot V_\mu$$

for every distribution μ, from which

$$\varrho^2 \leq M \cdot V(E)$$

or

$$C_\beta(E) \leq \left[\frac{M}{\varrho^2}\right]^{\frac{1}{\beta}}, \qquad (\beta = 1 - \alpha)$$

and so the first part of theorem 1 is proved.

2° If $f(\theta)$ is lower semi-continuous (i. e. $f(\theta)$ is the limit function of a non-decreasing sequence of continuous functions), the set E_ε where $f(\theta) > \varrho - \varepsilon$, $\varepsilon > 0$, is always open, from which by what has just been proved

$$C_\beta(E_\varepsilon) \leq \left[\frac{M}{(\varrho - \varepsilon)^2}\right]^{\frac{1}{\beta}},$$

and by (1. 8)

$$C_\beta(E) \leq \left[\frac{M}{\varrho^2}\right]^{\frac{1}{\beta}}. \qquad (\beta = 1 - \alpha)$$

Hence the second part of theorem 1 is proved.

CHAPTER 2

On the Total Variation on Radii of Harmonic Functions belonging to (S_α)

7. — Let $f(r, \theta)$ be a function, harmonic in the unit circle, and let $F(r, \theta)$ denote its total variation on the segment $(0, r)$ of the radius θ:

$$F(r, \theta) = \int_0^r \left| \frac{\partial}{\partial r} f(r, \theta) \right| dr. \tag{2.1}$$

For the sake of brevity we shall sometimes write

$$\frac{\partial F(r, \theta)}{\partial r} = \frac{\partial F}{\partial r} \quad \text{and} \quad F(1, \theta) = F(\theta) = F,$$

and we shall call this last function *the strong majorant.*

We now assert that the following theorem holds:

Theorem 2. *If the real function $f(r, \theta)$ is harmonic in the unit circle and belongs to the class (S_α), then its strong majorant $F(\theta)$ is a summable function, also belonging to (S_α), and*

$$S_\alpha(F) \leq M \cdot S_\alpha(f),$$

M depending only on α.

To prove this theorem we need a lemma.

8. — Subsequently we shall often use the short notation

$$\log \frac{1}{r} = \varphi(r) = \varphi.$$

Suppose that $g(r,\theta)$ is a real function, defined in the unit circle, and that $\frac{\partial g}{\partial r}$ and $\frac{\partial^2 g}{\partial r\,\partial\theta}$ exist at least almost everywhere. We shall also use the abbreviations[1]

$$D_\alpha(g) \equiv \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left(\frac{\partial g}{\partial r}\right)^2 \varphi^{1-\alpha} r\,d\theta\,dr \tag{2.2}$$

and

$$E_\alpha(g) \equiv \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left(\frac{\partial^2 g}{\partial r\,\partial\theta}\right)^2 \varphi^{3-\alpha} r\,d\theta\,dr. \tag{2.3}$$

Lemma 2. *If the real function $f(r,\theta)$ is harmonic in the unit circle and belongs to the class (S_α), then*

$$D_\alpha(f) = M\cdot S_\alpha(f) \tag{2.4}$$

and

$$E_\alpha(f) = M'\cdot S_\alpha(f), \tag{2.5}$$

where M and M' depend only on α.

Proof. We have that

$$\int_0^{2\pi}\left(\frac{\partial f}{\partial r}\right)^2 d\theta = \int_0^{2\pi}\left\{\sum_{n=1}^{\infty} n r^{n-1}(a_n\cos n\theta + b_n\sin n\theta)\right\}^2 d\theta =$$

$$= \int_0^{2\pi}\sum_{n=1}^{\infty} n^2 r^{2n-2}(a_n^2\cos^2 n\theta + b_n^2\sin^2 n\theta)\,d\theta$$

and by (2.2)

$$D_\alpha(f) = \int_0^1\sum_{n=1}^{\infty} n^2 r^{2n-1}(a_n^2+b_n^2)\varphi^{1-\alpha}\,dr. \tag{2.6}$$

The substitution $\log\frac{1}{r} = \frac{t}{n}$ gives

[1] If g is harmonic, then (2.2) reduces for $\alpha = 1$ to the DIRICHLET integral (irrespective of a constant factor): $\iint_{x^2+y^2<1}(g_x'^2+g_y'^2)\,dx\,dy = 2\int_0^{2\pi}\int_0^1 g_r'^2\,r\,dr\,d\theta$. In (2.2) the function $\varphi^{1-\alpha}$ serves as a convergence factor.

$$(2.7)\qquad \int_0^1 n^2 r^{2n-1}\left(\log\frac{1}{r}\right)^{1-\alpha} dr = n^\alpha \int_0^\infty e^{-2t} t^{1-\alpha}\, dt,$$

the integral being convergent and having a positive value. Our assertion (2.4) now follows from (2.6), (2.7), and (1.3).

Further we have that

$$\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial r\,\partial\theta}\right)^2 d\theta = \int_0^{2\pi}\left\{\sum_{n=1}^{\infty} n^2 r^{n-1}(-a_n \sin n\theta + b_n \cos n\theta)\right\}^2 d\theta$$

and by (2.3)

$$(2.8)\qquad E_\alpha(f) = \int_0^1 \sum_{n=1}^{\infty} n^4 r^{2n-1}(a_n^2 + b_n^2)\varphi^{3-\alpha}\, dr.$$

The same substitution as before gives

$$(2.9)\qquad \int_0^1 n^4 r^{2n-1}\left(\log\frac{1}{r}\right)^{3-\alpha} dr = n^\alpha \int_0^\infty e^{-2t} t^{3-\alpha}\, dt,$$

this integral also having a positive finite value. The formula (2.5) now follows from (2.8), (2.9), and (1.3), and the lemma is completely proved.

The above term-by-term integrations are legitimate, the series considered being uniformly convergent.

9. — *Proof of theorem 2.* Formula (2.1) gives that $F_r' = |f_r'|$, from which

$$(2.10)\qquad \left(\frac{\partial F}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2.$$

Furthermore we need to show that

$$(2.11)\qquad \left(\frac{\partial^2 F}{\partial r\,\partial\theta}\right)^2 = \left(\frac{\partial^2 f}{\partial r\,\partial\theta}\right)^2$$

almost everywhere in the unit circle. To this end observe first that f_r' and rf_r' are zero at the same points; we here disregard the origin throughout the whole argument. Further rf_r' is har-

monic and consequently continuous in the unit circle. If $rf_r' \neq 0$ at a point (r, θ), then rf_r' has a constant sign in a neighbourhood of (r, θ). Hence it follows that $F_{r\theta}'' = \frac{\partial}{\partial \theta}|f_r'| = \pm f_{r\theta}''$ at every point where $f_r' \neq 0$; here the plus sign holds if $f_r' > 0$ and the minus sign holds if $f_r' < 0$. Furthermore $f_{r\theta}''$ exists everywhere. Thus it is sufficient to show that $f_r' = 0$ at most in a set of measure zero, or that $rf_r' = 0$ at most in a set of measure zero. But this last proposition is evident: the points of any circle $|z| \leq r_0$, $r_0 < 1$, where a function, harmonic and not identically zero in the unit circle, vanishes consist of a finite number of analytic arcs.

The formulae (2.2), (2.3), (2.10), and (2.11) now give

$$D_\alpha(F) = D_\alpha(f) \quad \text{and} \quad E_\alpha(F) = E_\alpha(f). \tag{2.12}$$

That $F(\theta)$ is summable s seen by

$$\int_0^{2\pi} F(\theta)\, d\theta = \int_0^1 \int_0^{2\pi} \left| \frac{\partial f}{\partial r} \right| d\theta\, dr = \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1 = P + Q,$$

where P is convergent (viz. it is an integral of a bounded function), and also Q is convergent by Schwarz's inequality:

$$Q^2 \leq \int_{\frac{1}{2}}^1 \int_0^{2\pi} \left(\frac{\partial f}{\partial r}\right)^2 \varphi^{1-\alpha} r\, d\theta\, dr \cdot \int_{\frac{1}{2}}^1 \int_0^{2\pi} \frac{d\theta\, dr}{\varphi^{1-\alpha} r} <$$

$$< 2\pi^2 \cdot D_\alpha(f) \cdot \int_{\frac{1}{2}}^1 \frac{dr}{r \left(\log \frac{1}{r}\right)^{1-\alpha}}.$$

Hence we can write

$$F(r, \theta) \sim \tfrac{1}{2} A_0(r) + \sum_{n=1}^{\infty} (A_n(r) \cos n\theta + B_n(r) \sin n\theta),$$

where the Fourier coefficients $A_n(r)$ and $B_n(r)$ are well-determined even for $r = 1$. We now get by what has been stated about the derivatives $\frac{\partial F}{\partial r}$ and $\frac{\partial^2 F}{\partial r\, \partial \theta}$, letting an apostrophe denote differentiation with respect to r,

$$\frac{\partial F}{\partial r} \sim \tfrac{1}{2} A_0' + \sum_{n=1}^{\infty} (A_n' \cos n\theta + B_n' \sin n\theta)$$

and

$$\frac{\partial^2 F}{\partial r\, \partial \theta} \sim \sum_{n=1}^{\infty} (n B_n' \cos n\theta - n A_n' \sin n\theta).$$

Parseval's theorem then gives

$$\frac{1}{\pi} \int_0^{2\pi} \left(\frac{\partial F}{\partial r}\right)^2 d\theta = \tfrac{1}{2} A_0'^2 + \sum_{n=1}^{\infty} (A_n'^2 + B_n'^2)$$

and

$$\frac{1}{\pi} \int_0^{2\pi} \left(\frac{\partial^2 F}{\partial r\, \partial \theta}\right)^2 d\theta = \sum_{n=1}^{\infty} (n^2 A_n'^2 + n^2 B_n'^2).$$

By (2.12) and lemma 2 we now get

$$(2.13) \qquad \int_0^1 \left\{ \tfrac{1}{2} A_0'^2 + \sum_{n=1}^{\infty} (A_n'^2 + B_n'^2)\,(1 + n^2 \varphi^2) \right\} \varphi^{1-\alpha} r\, dr =$$

$$= D_\alpha(F) + E_\alpha(F) = D_\alpha(f) + E_\alpha(f) = M \cdot S_\alpha(f)$$

where M has a new value. As $A_n(0) = 0$ for every n, we have

$$A_n(1) = \int_0^1 A_n'(r)\, dr,$$

and Schwarz's inequality yields

$$(2.14) \qquad A_n^2(1) \leq \left(\int_0^1 |A_n'(r)|\, dr \right)^2 \leq$$

$$\leq \int_0^1 A_n'^2 (1 + n^2 \varphi^2) \varphi^{1-\alpha} r\, dr \cdot \int_0^1 \frac{dr}{(1 + n^2 \varphi^2) \varphi^{1-\alpha} r}.$$

The substitution $\log \dfrac{1}{r} = \dfrac{t}{n}$ gives

$$(2.15)\qquad \int_0^1 \frac{dr}{\left(1+n^2\log^2\frac{1}{r}\right)\left(\log\frac{1}{r}\right)^{1-\alpha} r} = \frac{1}{n^\alpha}\int_0^\infty \frac{dt}{(1+t^2)\,t^{1-\alpha}} = \frac{M}{n^\alpha},$$

where M is a positive constant, depending only on α. The formulae (2.14) and (2.15) give

$$(2.16)\qquad n^\alpha \cdot A_n^2(1) \leq M \cdot \int_0^1 A_n'^2 (1+n^2\varphi^2)\,\varphi^{1-\alpha}\, r\, dr,$$

and quite analogously we get

$$(2.17)\qquad n^\alpha \cdot B_n^2(1) \leq M \cdot \int_0^1 B_n'^2 (1+n^2\varphi^2)\,\varphi^{1-\alpha}\, r\, dr.$$

From (2.16), (2.17), and (2.13) there follows

$$\sum_{n=1}^{\infty} n^\alpha (A_n^2(1) + B_n^2(1)) \leq M \cdot S_\alpha(f),$$

where M has a new value, and reference to (1.3) gives theorem 2.

10. — *Theorem 2 can be generalized to hold for complex harmonic functions* (e. g. for analytic functions, regular in the unit circle).

The definition (1.2) of the class (S_α) is then conveniently replaced by

$$S_\alpha(f) \equiv \int_0^\infty \int_0^{2\pi} \frac{|f(\theta+t) - f(\theta-t)|^2}{t^{1+\alpha}}\, d\theta\, dt < \infty.$$

The formula (1.3) then takes the following form

$$(2.18)\qquad S_\alpha(f) \equiv M \cdot \sum_{n=1}^{\infty} n^\alpha (|a_n|^2 + |b_n|^2),$$

M having the same value as in (1.3).

The only essential change in the proof of the generalized theorem 2 will be that (2.11) is replaced by:

$$(2.19)\qquad \left(\frac{\partial^2 F}{\partial r\,\partial\theta}\right)^2 \leq \left|\frac{\partial^2 f}{\partial r\,\partial\theta}\right|^2$$

holds almost everywhere. To show (2.19) we put $f = u + iv$, where u and v are real harmonic functions. Then

$$F''_{r\theta} = \frac{\partial}{\partial\theta}|f'_r| = \frac{\partial}{\partial\theta}\sqrt{u'^2_r + v'^2_r} = \frac{u'_r u''_{r\theta} + v'_r v''_{r\theta}}{|f'_r|}$$

and $F''_{r\theta}$ exists at all points where $f'_r \neq 0$, thus almost everywhere. Furthermore

$$\left|\frac{\partial}{\partial\theta}|f'_r|\right| \leq \left|\frac{\partial}{\partial\theta}f'_r\right|,$$

by the definition of the derivative and the relation $||a| - |b|| \leq |a - b|$, which holds for any complex numbers a and b. Thus we get

$$|F''_{r\theta}| = \left|\frac{\partial}{\partial\theta}|f'_r|\right| \leq \left|\frac{\partial}{\partial\theta}f'_r\right| = |f''_{r\theta}|,$$

and (2.19) follows.

11. — An immediate consequence of theorems 1 and 2 is the following proposition:

1° *If the function $f(r, \theta)$ is harmonic in the unit circle and belongs to (S_α), then its total variation*

$$\int_0^1 \left|\frac{\partial}{\partial r}f(r, \theta)\right| dr$$

is finite everywhere except, at most, on a set E on the unit circle for which

$$C_{1-\alpha}(E) = 0.$$

2° *Under the same conditions, the same proposition holds concerning the existence of the radial limit*

$$\lim_{r=1-0} f(r, \theta).$$

Item 2° obviously being an immediate consequence of item 1°, it is sufficient to prove the latter. By theorem 2 the strong majorant $F(\theta)$ belongs also to the class (S_α). Further $F(\theta)$ is lower semi-continuous (it is for example the limit function of the

sequence $F\left(1-\frac{1}{n},\ \theta\right)$, $n=1, 2, \ldots$). The set on the circumference of the unit circle, where the strong majorant is infinite, must be a subset of the set E_ϱ, where $F(\theta) > \varrho$, for every ϱ. Now, by theorem 1, the set E_ϱ being open,

$$\lim_{\varrho \infty} C_{1-\alpha}(E_\varrho) = 0,$$

and our proposition follows from (1. 8).

The above result 2°, however, is an immediate corollary of theorem 3. 1° of the next chapter. On the other hand it ought to be observed that our above proposition 1° does not follow from theorem 3. 1° (the radial limit can very well exist at a boundary point θ without the strong majorant there being finite). Thus the above result 1° has an interest of its own. Furthermore it has applications to the theory of analytic functions. We shall, however, not take up these questions here.

CHAPTER 3

On Singular Sets for Functions belonging to (S_α)

12. — We assert that the following theorem holds.

Theorem 3. *Consider a (real or complex) function $f(\theta)$, belonging to the class (S_α), and its Fourier series*

$$f(\theta) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

At every point θ, not belonging to a certain exceptional set E whose outer capacity of order $1-\alpha$ is zero, it holds 1° that the series converges and 2° that the primitive function of $f(\theta)$ has a determined derivative, equal to the sum of the series.

To prove this theorem we shall need some lemmas.

13. — We suppose that $a_0 = 0$ which is no essential restriction, and hence we write:

$$f(\theta) \sim \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{3.1}$$

We shall associate with the function $f(\theta)$ three new functions $g(\theta)$, $G(\theta)$, and $F(\theta)$ by the following conventions.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{A_n(\gamma)} (a_n \cos n\theta + b_n \sin n\theta), \tag{3.2}$$

where A_n is defined in lemma 6, page 43, $\gamma = 1 - \frac{\alpha}{2}$ by (5.1), and the coefficients a_n, b_n are the same as in (3.1). For the absolute values of the squares of the coefficients we have by lemma 6 and formula (2.18) that:

$$\sum_{n=1}^{\infty} \frac{|a_n|^2+|b_n|^2}{A_n^2(\gamma)} \leq \frac{1}{m^2} \sum_{n=1}^{\infty} n^{2(1-\alpha)}(|a_n|^2+|b_n|^2) =$$

$$= \frac{1}{m^2} \sum_{n=1}^{\infty} n^{\alpha}(|a_n|^2+|b_n|^2) < \infty .$$

By the Riesz-Fischer theorem, (3.2) then is the Fourier series of a certain function $g(\theta)$ belonging to the class L^2:

$$g(\theta) \sim \sum_{n=1}^{\infty} \frac{1}{A_n(\gamma)}(a_n \cos n\theta + b_n \sin n\theta). \tag{3.3}$$

Observe that if we expand the integral

$$\frac{1}{\pi} \int_0^{2\pi} g(t)\, h_\gamma(t-\theta)\, dt \tag{3.4}$$

or (compare lemma 8, page 45)

$$\frac{1}{\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{A_n(\gamma)}(a_n \cos nt + b_n \sin nt) \right\} \left\{ 1 + \sum_{n=1}^{\infty} A_n(\gamma) \cos n(t-\theta) \right\} dt$$

in a Fourier series in θ, then we obtain again the series (3.1). Let us now put

$$G(t) = |g(t)| \tag{3.5}$$

and

$$F(\theta) = \frac{1}{\pi} \int_0^{2\pi} G(t)\, h_\gamma(t-\theta)\, dt. \tag{3.6}$$

The function h_γ is positive everywhere (compare the proof of lemma 9.1°). Thus, $F(\theta)$ is a non-negative function, having a finite determined value at every point θ where the function $g(t) \cdot h_\gamma(t-\theta)$ is summable and the value $+\infty$ at every point where $g(t) \cdot h_\gamma(t-\theta)$ is not summable. We can now give one of our lemmas.

Lemma 3. *Suppose that $f(\theta)$ belongs to the class (S_α) and is normalized by the conditions $S_\alpha(f) = 1$ and $a_0 = 0$. We then have for the set E_ϱ where its corresponding function $F(\theta) > \varrho > 0$, that*

$$C_{1-\alpha}(E_\varrho) \leq M \cdot \varrho^{-\frac{2}{1-\alpha}},$$

M depending only on α.

Proof. The function $F(\theta)$ is the limit function of the non-decreasing sequence of continuous functions

$$F_n(\theta) = \frac{1}{\pi} \int_{-\frac{1}{n}}^{2\pi - \frac{1}{n}} G(t+\theta)\, h_\gamma(t)\, dt. \qquad (n = 1, 2, \ldots)$$

Hence it follows that $F(\theta)$ is lower semi-continuous and the set E_ϱ is open.

Let μ be a positive unit mass, distributed on E_ϱ, and let V_μ be the least upper bound of the corresponding potential of order β. The definition of E_ϱ and (3.6) give

$$\varrho \leq \int_0^{2\pi} F(\theta)\, d\mu(\theta) = \int_0^{2\pi} \frac{1}{\pi} \int_0^{2\pi} G(t)\, h_\gamma(t-\theta)\, dt\, d\mu(\theta).$$

Since the integrand is non-negative, the order of integration can be reversed:

$$\varrho \leq \frac{1}{\pi} \int_0^{2\pi} G(t)\, H(t)\, dt,$$

where

$$H(t) = \int_0^{2\pi} h_\gamma(t-\theta)\, d\mu(\theta). \tag{3.7}$$

Schwarz's inequality then yields

$$\varrho^2 \leq \frac{1}{\pi} \int_0^{2\pi} G^2(t)\, dt \cdot \frac{1}{\pi} \int_0^{2\pi} H^2(t)\, dt. \tag{3.8}$$

To estimate the first integral of (3.8) we use Parseval's theorem and pay attention to the normalizations and the relations (3.5), (3.3), and (2.18):

$$\frac{1}{\pi} \int_0^{2\pi} G^2(t)\, dt = \frac{1}{\pi} \int_0^{2\pi} |g(t)|^2\, dt = \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{A_n^2(\gamma)} \leq \tag{3.9}$$

$$\leq \frac{1}{m^2} \sum_{n=1}^{\infty} n^\alpha (|a_n|^2 + |b_n|^2) = \frac{1}{m^2 M} S_\alpha(f) = M,$$

M having a new value.

To estimate the second integral of (3.8) we reverse once more the order of integration, the integrand being positive, and take account of (3.7), lemma 10, page 49, and lemma 7, page 44:

$$(3.10)\qquad \int_0^{2\pi} H^2(t)\,dt = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} h_\gamma(t-x)\,h_\gamma(t-y)\,d\mu(x)\,d\mu(y)\,dt \leq$$

$$\leq \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{|e^{it}-e^{ix}|^\gamma}\cdot\frac{1}{|e^{it}-e^{iy}|^\gamma}\,d\mu(x)\,d\mu(y)\,dt \leq$$

$$\leq M\int_0^{2\pi} d\mu(y)\int_0^{2\pi}\frac{d\mu(x)}{|e^{ix}-e^{iy}|^\beta}.$$

where M has the same value as in lemma 10. The definition of V_μ implies that

$$\int_0^{2\pi}\frac{d\mu(x)}{|e^{ix}-e^{iy}|^\beta} \leq V_\mu,$$

and hence by (3.10)

$$(3.11)\qquad \int_0^{2\pi} H^2(t)\,dt \leq M\cdot V_\mu.$$

The formulae (3.8), (3.9), and (3.11) now give

$$\varrho^2 \leq M\cdot V_\mu,$$

once more with a new value of M. Consequently for $V(E_\varrho) = \inf_\mu V_\mu$ it follows that

$$\varrho^2 \leq M\cdot V(E_\varrho),$$

and the definition of capacity together with the fact that E_ϱ is open gives

$$C_\beta(E_\varrho) \leq \left(\frac{M}{\varrho^2}\right)^{\frac{1}{\beta}}$$

and hence lemma 3 is proved.

14. — Let $g(\theta)$ retain its significance from formula (3.3). We then can formulate a new lemma.

Lemma 4. *The function*

$$f^*(\theta) = \frac{1}{\pi} \int_0^{2\pi} g(t)\, h_\gamma(t-\theta)\, dt \tag{3.12}$$

has a finite and determined value everywhere except perhaps on a set E for which

$$\bar{C}_{1-\alpha}(E) = 0.$$

Before the proof we make the following remark. At every point θ where the integrand of (3.12) is summable we define $f^*(\theta)$ by the relation (3.12). At the remaining points θ we let the value of $f^*(\theta)$ be undefined.

Proof. The set E where the integrand of (3.12) is not summable must by (3.5) and (3.6) be a subset of the set E_ϱ where $F(\theta) > \varrho$ for every ϱ. And lemma 3 yields

$$\lim_{\varrho=\infty} C_\beta(E_\varrho) = 0,$$

(even if $f(\theta)$ is not normalized), and hence our proposition follows from (1.8).

(It ought to be observed that the functions $f(\theta)$ and $f^*(\theta)$ have identical Fourier series and, consequently, that they are equal except perhaps on a set of measure zero.)

15. — *Proof of theorem 3.* Throughout the whole proof let θ be any fixed point, at which $f^*(\theta)$ has a finite determined value. Lemma 4 shows that we exclude from our arguments only a set E with $\bar{C}_{1-\alpha}(E) = 0$.

1° Denote by $f_n(\theta)$ the following partial sum of the series (3.1)

$$f_n(\theta) = \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta).$$

And let $h_{\gamma,n}$ have the same significance as in lemma 8, page 45.

Then we have that

$$f_n(\theta) = \frac{1}{\pi} \int_0^{2\pi} g(t)\, h_{\gamma,n}(t-\theta)\, dt.$$

Formula (3.12) now gives

$$(3.13)\quad f^*(\theta)-f_n(\theta)=\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_\gamma(t)\,dt-\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_{\gamma,n}(t)\,dt=$$

$$=\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_\gamma(t)\left(1-\frac{h_{\gamma,n}(t)}{h_\gamma(t)}\right)dt.$$

In this last integral the first factor $g(t+\theta)\,h_\gamma(t)$ is a summable function by assumption, and the second factor is uniformly bounded and converges to zero for every $\theta \neq 0$ and $\neq 2\pi$ by lemma 8. Hence the integral referred to converges to zero as $n \to \infty$ by LEBESGUE's general convergence theorem, and the first part of theorem 3 is proved.

2° Let $f_\delta(\theta)$ and $h_{\gamma,\delta}(\theta)$ denote the quantities

$$f_\delta(\theta)=\frac{1}{\delta}\int_\theta^{\theta+\delta} f(\theta)\,d\theta \quad\text{and}\quad h_{\gamma,\delta}(\theta)=\frac{1}{\delta}\int_\theta^{\theta+\delta} h_\gamma(\theta)\,d\theta,$$

where δ is a positive or negative number.

Then we have (compare the last comment on the proof of lemma 4) by (3.12)

$$f_\delta(\theta)=\frac{1}{\delta}\int_\theta^{\theta+\delta} f^*(\theta)\,d\theta=\frac{1}{\delta}\int_\theta^{\theta+\delta}\left\{\frac{1}{\pi}\int_0^{2\pi} g(t)\,h_\gamma(t-\theta)\,dt\right\}d\theta=$$

$$=\frac{1}{\pi}\int_0^{2\pi} g(t)\left\{\frac{1}{\delta}\int_\theta^{\theta+\delta} h_\gamma(t-\theta)\,d\theta\right\}dt=\frac{1}{\pi}\int_0^{2\pi} g(t)\,h_{\gamma,-\delta}(t-\theta)\,dt=$$

$$=\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_{\gamma,-\delta}(t)\,dt.$$

We then obtain

$$(3.14)\quad f^*(\theta)-f_\delta(\theta)=\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_\gamma(t)\,dt-$$

$$-\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_{\gamma,-\delta}(t)\,dt=\frac{1}{\pi}\int_0^{2\pi} g(t+\theta)\,h_\gamma(t)\left(1-\frac{h_{\gamma,-\delta}(t)}{h_\gamma(t)}\right)dt.$$

In this last integral the factor $g(t+\theta)h_\gamma(t)$ is summable by assumption and the last factor is uniformly bounded and converges to zero for every $t \neq 0$ and $\neq 2\pi$ by lemma 9, page 47. Hence the above integral converges to zero as $\delta \to 0$ by LEBESGUE's general convergence theorem, and if we finally observe that $f^*(\theta)$ has the same value in (3.13) and in (3.14) the second part of theorem 3 is completely proved.

16. — By means of our lemma 3 we can also prove a theorem on *uniform* convergence.

Theorem 4. *Suppose that the function $f(\theta)$ belongs to the class (S_α), and that ε is an arbitrary positive number less than α. Then there is an open set $O = O_\varepsilon$ with $C_{1-\alpha+\varepsilon}(O) < \varepsilon$ such that the Fourier series of $f(\theta)$ converges uniformly outside O.*

Proof. We shall use the notations

$$\gamma = 1 - \frac{\alpha}{2} \text{ and } \bar{\gamma} = 1 - \frac{\alpha - \varepsilon}{2}$$

and by analogy to (3.6) the notation

$$\bar{F}(\theta) = \frac{1}{\pi}\int_0^{2\pi} G(t)\, h_{\bar{\gamma}}(t-\theta)\, dt.$$

It follows from (5.9), page 48, that

$$\lim_{t=0} h_\gamma(t) \cdot |t|^\gamma = \text{const.} > 0,$$

and hence

$$\lim_{t=0} \frac{h_{\bar{\gamma}}(t)}{h_\gamma(t)} = \infty. \tag{3.15}$$

Further we have for $0 < \varepsilon \leq \varepsilon_0 < \alpha$ that

$$h_{\bar{\gamma}}(t) \geq m \cdot h_\gamma(t) \tag{3.16}$$

everywhere, m being a positive constant depending only on α and ε_0; in fact, it is easily shown by (5.9) that $m > 2^{-\frac{1}{2}\varepsilon_0} \cdot \cos\left(1 - \frac{\alpha - \varepsilon_0}{2}\right) \cdot \frac{\pi}{2}$.

It will be seen that (3.15) and (3.16) are the only properties of h_γ relevant for theorem 4.

If the function $f(\theta)$ belongs to the class (S_α), it belongs evidently also to the class $(S_{\alpha-\varepsilon})$. Suppose $S_\alpha(f) = 1$ and $a_0 = 0$, which is no essential restriction. Hence $S_{\alpha-\varepsilon}(f)$ is finite, and by lemma 3 we can choose ϱ so large that the open set O, where $\bar{F}(\theta) > m \cdot \varrho$, satisfies the condition $C_{1-\alpha+\varepsilon}(O) < \varepsilon$. At every point θ of the complementary set F of O it then holds that

$$\bar{F}(\theta) = \frac{1}{\pi}\int_0^{2\pi} G(t)\, h_{\bar{\gamma}}(t-\theta)\, dt \leq m \cdot \varrho \tag{3.17}$$

and by (3.16)

$$F(\theta) = \frac{1}{\pi}\int_0^{2\pi} G(t)\, h_\gamma(t-\theta)\, dt \leq \varrho. \tag{3.18}$$

Take $\eta > 0$ arbitrarily small. By the formulae (3.13) and (3.5) we have in F that

$$|f^*(\theta) - f_n(\theta)| \leq \frac{1}{\pi}\left[\int_{-\delta}^{\delta} + \int_{\delta}^{2\pi-\delta}\right] G(t+\theta)\, h_\gamma(t)\left|1 - \frac{h_{\gamma,n}(t)}{h_\gamma(t)}\right| dt = P + Q.$$

$$(0 < \delta < \pi)$$

By lemma 8.1°, page 45, we have independently of n

$$P \leq M\int_{-\delta}^{\delta} G(t+\theta)\, h_\gamma(t)\, dt,$$

where M is a constant depending only on α. Take $\delta > 0$ so small that by (3.15)

$$\frac{h_{\bar{\gamma}}(t)}{h_\gamma(t)} \leq \frac{\eta}{2M\bar{m}\varrho} \quad \text{for } |t| \leq \delta.$$

Hence by (3.17)

$$P \leq M\int_{-\delta}^{\delta} G(t+\theta)\cdot h_{\bar{\gamma}}(t)\cdot\frac{h_\gamma(t)}{h_{\bar{\gamma}}(t)}\, dt \leq \frac{\eta}{2}.$$

Choose then n_0 so large that by lemma 8.2°

$$\left| 1 - \frac{h_{\gamma, n}(t)}{h_\gamma(t)} \right| \leq \frac{\eta}{2\varrho} \text{ for } n \geq n_0 \text{ and } \delta \leq t \leq 2\pi - \delta.$$

Hence by (3.18)

$$Q \leq \frac{1}{\pi} \int_{\delta}^{2\pi-\delta} G(t+\theta)\, h_\gamma(t) \cdot \frac{\eta}{2\varrho}\, dt \leq \frac{\eta}{2}$$

for $n \geq n_0$. Thus,

$$|f^*(\theta) - f_n(\theta)| \leq \eta$$

for n sufficiently large and independently of where θ is given in F, and theorem 4 is proved.

CHAPTER 4

On the Unicity Sets of Trigonometrical Series belonging to (T_α)

17. — Consider the trigonometrical series (with real or complex coefficients)

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta), \tag{4.1}$$

belonging to a certain class (T) of trigonometrical series, defined by some condition imposed on the coefficients a_n, b_n, implying that the function

$$f(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta) \tag{4.2}$$

is regular for $r < 1$.

The series (4.1) is said to be *summable by the* Abel-Poisson *method* at every point θ for which

$$\lim_{r=1-0} f(r,\theta)$$

exists.

We shall say that a set $E < (0, 2\pi)$ is a *unicity set* for the class (T) with respect to the Abel-Poisson summability method or shorter a *U-set*, if all trigonometrical series, belonging to (T) and Abel-Poisson summable to zero outside E, are identically zero.

On the contrary we shall say that E is a *multiplicity set* for the class (T) with respect to the Abel-Poisson summability method or shorter an *M-set*, if there exists a trigonometrical series belonging to (T) and Abel-Poisson summable to zero outside E without being identically zero.

By the class (T_α) of trigonometrical series we shall understand, as already mentioned in the introduction, all series (4. 1) for which

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{n^\alpha}$$

is convergent. Here α is a constant, $0 < \alpha < 1$.

18. — We now assert that the following theorem holds:

Theorem 5. *A closed set E is a U-set or an M-set with respect to the class (T_α) according as its capacity of order $1 - \alpha$ is zero or positive.*

To prove this theorem[1] we shall need a lemma.

19. — Let the set E_k consist of k disjoint closed arcs of the unit circle. Let μ be the equilibrium distribution of the mass $+1$ on E_k, let $u(1, \theta)$ be the corresponding equilibrium potential of order $\beta = 1 - \alpha$:

$$u(1, \theta) = \int_0^{2\pi} \frac{d\mu(\varphi)}{|e^{i\theta} - e^{i\varphi}|^\beta}, \tag{4.3}$$

and let $V(E_k)$ be its least upper bound. Then

$$u(1, \theta) = V(E_k) \tag{4.4}$$

everywhere on E_k (compare section 3).

Furthermore $u(1, \theta)$ is continuous. Let

$$\frac{1}{|1 - e^{i(\theta - \varphi)}|^\beta} = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos n(\theta - \varphi) \tag{4.5}$$

be the Fourier series of the kernel. Hence we now have

$$u(1, \theta) \sim \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n (h_n \cos n\theta + k_n \sin n\theta), \tag{4.6}$$

where the numbers A_n are defined in (4. 5) and the numbers h_n, k_n in (1. 9). (To obtain this, multiply (4. 3) by $\cos n\theta$ or $\sin n\theta$

[1] The method of proof used here is based on that used by BEURLING in the case $\alpha = 1$ in the lecture cited on page 8.

and inverse the order of integration, which is evidently legitimate in this case.)

Lemma 5. *The coefficients* A_n *of* (4.6) *satisfy the condition*

$$\lim_{n=\infty} A_n \cdot n^{\alpha} = \text{const.} > 0, \qquad (\alpha = 1 - \beta) \tag{4.7}$$

where the value of the constant depends only on α.

Proof. The function

$$\varphi(\theta) = \frac{1}{|1 - e^{i\theta}|^{\beta}} - \frac{1}{\theta^{\beta}}$$

is of bounded variation in the interval $0 \leq \theta \leq \pi$ (observe that $\lim_{\theta=0} \varphi(\theta) = 0$), and hence its Fourier coefficients are $O\left(\frac{1}{n}\right)$. Consequently it is sufficient to show the relation (4.7) for the Fourier coefficients of the even periodic function which in $(0, \pi)$ coincides with $\theta^{-\beta}$. For these last Fourier coefficients the substitution $n\theta = t$ gives

$$\frac{2}{\pi}\int_0^{\pi} \theta^{-\beta} \cos n\theta \, d\theta = \frac{2}{\pi}\frac{1}{n^{1-\beta}}\int_0^{n\pi} t^{-\beta} \cos t \, dt,$$

and the constant in (4.7) is equal to

$$\frac{2}{\pi}\int_0^{\infty} t^{-\beta} \cos t \, dt,$$

which is certainly positive. Hence lemma 5 is proved.

20. — *Proof of the first part of theorem 5:* if

$$C_{1-\alpha}(E) = 0, \tag{4.8}$$

then E is a U-set, i. e. (4.1) vanishes identically.

1° We shall first prove that $a_0 = 0$ and, to this end, we can assume, without loss of generality, the following normalization:

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{n^{\alpha}} = 1.$$

The complement of E is an open set, say $\sum_{\nu=1}^{\infty} \omega_\nu$, where the ω_ν's are disjoint open arcs.

The complement of $\sum_{\nu=1}^{k} \omega_\nu$ consists of k disjoint closed arcs, say $\alpha_\nu = \alpha_{\nu,k}$, where $\nu = 1, 2, \ldots, k$. (Some of them degenerate perhaps to points.)[1] Then $\sum_{\nu=1}^{k} \alpha_\nu > E$.

Enclose for a certain k each particular α_ν in a closed arc $\beta_\nu = \beta_{\nu,k}$. Here the end points of α_ν are to be inner points of β_ν, and furthermore the arcs β_ν are to be disjoint. We set $\sum_{\nu=1}^{k} \beta_\nu = E_k$ and, finally, we choose the arcs β_ν so that

$$\left.\begin{aligned} &E_1 > E_2 > \cdots > E \\ &E_1 \cdot E_2 \cdot \cdots = E \end{aligned}\right\}. \tag{4.9}$$

It is obvious that this can always be done.

We have that

$$\pi a_0 = \int_0^{2\pi} f(r, \theta)\, d\theta = \int_{E_k} + \int_{U - E_k},$$

where U denotes the whole circumference. By an important theorem of VERBLUNSKY[2] the last integral tends to zero as $r \to 1 - 0$. Letting $\varepsilon(r)$, $\varepsilon_1(r)$, ... denote quantities, tending to zero as $r \to 1 - 0$, we can consequently write

$$\pi a_0 + \varepsilon(r) = \int_{E_k} f(r, \theta)\, d\theta$$

or by (4.4)

[1] In fact it can be proved that this case does not occur. Thus, the word "closed" in the formulation of theorem 5 can be replaced by "perfect".

[2] If a_n, $b_n = o(n)$ and if $\lim_{r=1-0} f(r, \theta) = 0$ at every point of an open arc ω, then the same relation holds *uniformly* in every closed subarc of ω. Compare [6], page 379, theorem IV.

$$\pi a_0 + \varepsilon(r) = \frac{1}{V(E_k)} \int_{E_k} f(r,\theta) \cdot u(1,\theta)\, d\theta = \frac{1}{V(E_k)} \left[\int_0^{2\pi} - \int_{U-E_k} \right].$$

One more application of VERBLUNSKY's theorem just cited now gives

$$\pi a_0 + \varepsilon_1(r) = \frac{1}{V(E_k)} \int_0^{2\pi} f(r,\theta) \cdot u(1,\theta)\, d\theta.$$

The formulae (4.2) and (4.6) yield

$$a_0 + \varepsilon_2(r) = \frac{1}{\pi V(E_k)} \int_0^{2\pi} \left\{ \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \right\} \cdot$$

$$\cdot \left\{ \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n (h_n \cos n\theta + k_n \sin n\theta) \right\} d\theta =$$

$$= \frac{a_0 A_0}{2 V(E_k)} + \frac{1}{V(E_k)} \sum_{n=1}^{\infty} r^n A_n (a_n h_n + b_n k_n) = P + Q$$

by PARSEVAL's theorem, the functions f and u being continuous. SCHWARZ's inequality and our normalization yield:

$$|Q|^2 \leq \frac{1}{V^2(E_k)} \sum_{n=1}^{\infty} r^{2n} \frac{|a_n|^2 + |b_n|^2}{n^\alpha} \cdot \sum_{n=1}^{\infty} n^\alpha A_n^2 (h_n^2 + k_n^2) <$$

$$< \frac{1}{V^2(E_k)} \cdot \sum_{n=1}^{\infty} n^\alpha A_n^2 (h_n^2 + k_n^2).$$

From lemma 5 and formula (1.10) then follows

$$|Q|^2 \leq \frac{M^2}{V(E_k)},$$

M having a new value, and we get

$$|a_0| \leq \frac{|a_0| A_0}{2 V(E_k)} + \frac{M}{\sqrt{V(E_k)}}.$$

By (4.8), (4.9), and a known property of sequences of potential functions (compare FROSTMAN, [2], § 8; see also § 3: the selection theorem) we have that $V(E_k) \to \infty$ as $k \to \infty$ and $a_0 = 0$ follows.

2° To prove that also the remaining coefficients of (4. 1) vanish, we write (4. 1) in its complex form

$$\sum_{m=-\infty}^{\infty} c_m e^{im\theta} \tag{4.10}$$

where for $n \geq 0$

$$2c_n = a_n - ib_n \quad \text{and} \quad 2c_{-n} = a_n + ib_n. \tag{4.11}$$

It is easily seen that

$$|a_n|^2 + |b_n|^2 = 2|c_n|^2 + 2|c_{-n}|^2, \qquad (n \geq 0)$$

and hence it follows that we, if we wish, can characterize the class (T_α) by the condition

$$\sum_{m=-\infty}^{\infty}{}' \frac{|c_m|^2}{|m|^\alpha} < \infty \tag{4.12}$$

(the apostrophe denoting that the term $m = 0$ is missing).

We multiply the series (4. 10) by $e^{-in\theta}$ (n is a positive integer):

$$\sum_{m=-\infty}^{\infty} c_m e^{i(m-n)\theta}. \tag{4.13}$$

This new trigonometrical series belongs also to the class (T_α), for (letting an apostrophe denote that the term $m = n$ is missing) we have

$$\sum_{m=-\infty}^{\infty}{}' \frac{|c_m|^2}{|m-n|^\alpha} = \sum_{-\infty}^{-1} + \sum_{0}^{2n-1}{}' + \sum_{2n}^{\infty} = S_1 + S_2 + S_3,$$

where by (4. 12)

$$S_1 \leq \sum_{-\infty}^{-1} \frac{|c_m|^2}{|m|^\alpha} < \infty,$$

$$S_2 < \infty,$$

and

$$S_3 \leq \sum_{2n}^{\infty} \left(\frac{2}{m}\right)^\alpha |c_m|^2 < \infty.$$

The function

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} c_m r^{|m|} e^{im\theta}$$

tends to zero for $r \to 1-0$ everywhere outside E by assumption. We need also to show that the function corresponding to the series (4.13)

$$\varphi(r,\theta) = \sum_{m=-\infty}^{\infty} c_m r^{|m-n|} e^{i(m-n)\theta} = \sum_{-\infty}^{-1} + \sum_{0}^{n-1} + \sum_{n}^{\infty} = P + Q + R$$

satisfies the condition

$$\lim_{r=1-0} \varphi(r,\theta) = 0 \tag{4.14}$$

in every point θ outside E. To this end observe that for $r \to 1-0$

$$r^n \text{ and } r^{-n} = 1 + O(1-r) \tag{4.15}$$

and that by SCHWARZ's inequality, the condition (4.12), and an argument analogous to lemma 6

$$\sum_{-\infty}^{\infty}{}' |c_m| r^{|m|} \leq \sqrt{\sum{}' \frac{|c_m|^2}{|m|^\alpha} \cdot \sum{}' |m|^\alpha r^{2|m|}} = \tag{4.16}$$

$$= \sqrt{O\left(\frac{1}{1-r^2}\right)^{1+\alpha}} = O\left(\frac{1}{1-r^2}\right)^{\frac{1+\alpha}{2}} = O(1-r)^{-\frac{1+\alpha}{2}}.$$

Formulae (4.15) and (4.16) give

$$e^{in\theta} \cdot P = r^n \cdot \sum_{-\infty}^{-1} c_m r^{|m|} e^{im\theta} = \sum_{-\infty}^{-1} c_m r^{|m|} e^{im\theta} + O(1-r)^{\frac{1-\alpha}{2}},$$

$$e^{in\theta} \cdot Q = r^n \cdot \sum_{0}^{n-1} c_m r^{-m} e^{im\theta} = \sum_{0}^{n-1} c_m r^m e^{im\theta} + O(1-r),$$

and

$$e^{in\theta} \cdot R = r^{-n} \cdot \sum_{n}^{\infty} c_m r^m e^{im\theta} = \sum_{n}^{\infty} c_m r^m e^{im\theta} + O(1-r)^{\frac{1-\alpha}{2}},$$

from which

$$\varphi(r,\theta) = e^{-in\theta} \cdot f(r,\theta) + O(1-r)^{\frac{1-\alpha}{2}} \tag{4.17}$$

and (4.14) follows everywhere outside E.[1]

[1] It is easily seen that O in (4.17) can be replaced by o, if we cut off a suitable partial sum in the series (4.16).

We can now apply the argument of item 1° to the series (4.13), and we find that $c_n = 0$ $(n = 1, 2, \ldots)$. Quite analogously we get $c_{-n} = 0$ $(n = 1, 2, \ldots)$, and (4.11) gives that all the coefficients a_n, b_n $(n = 1, 2, \ldots)$ vanish. Hence the first part of theorem 5 is proved.

21. — *Proof of the second part of theorem 5:* if

$$C_{1-\alpha}(E) > 0,$$

then E is an M-set, i. e. (4.1) does not necessarily vanish identically.

Let μ be a distribution of the unit mass on E, such that V_μ is finite. Hence by lemma 1

$$\sum_{n=1}^{\infty} \frac{h_n^2 + k_n^2}{n^\alpha} < \infty. \tag{4.18}$$

Let $P(r, \theta)$ denote Poisson's kernel

$$P(r, \theta) = \frac{1}{2} \cdot \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta,$$

and consider the function

$$f(r, \theta) = \int_0^{2\pi} P(r, \theta - \varphi)\, d\mu(\varphi) \sim \frac{1}{2} + \sum_{n=1}^{\infty} r^n (h_n \cos n\theta + k_n \sin n\theta).$$

It satisfies the condition

$$\lim_{r=1-0} f(r, \theta) = 0$$

everywhere outside E and, as (4.18) holds, the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} (h_n \cos n\theta + k_n \sin n\theta)$$

furnishes the required example.

CHAPTER 5

Lemmas concerning an Elementary Function

22. — In the preceding chapters we have frequent use for certain properties of the elementary function $w = \left(\frac{1}{1-z}\right)^{\beta}$, where $0 < \beta < 1$ and $|z| \leq 1$. Here we shall always consider the principal branch of the function (i. e. the branch taking the value $w = 1$ for $z = 0$), even if it is not pointed out every time. This chapter is devoted to a derivation of the mentioned properties. First we lay down some notational conventions.

The symbol α means everywhere in this paper a constant number, $0 < \alpha < 1$. The symbols β and γ are always used with the meanings

$$\beta = 1 - \alpha \quad \text{and} \quad \gamma = 1 - \frac{\alpha}{2}. \tag{5.1}$$

Hence they are constants, $0 < \beta < 1$ and $\frac{1}{2} < \gamma < 1$.

The letters m and M (and sometimes M') always denote *positive* constants whose values depend only upon α (or β or γ). They can very well have different values in different places.

23. — **Lemma 6.** *Consider the function*

$$h_\beta(r, \theta) = R\left(\frac{1}{1 - re^{i\theta}}\right)^{\beta},$$

where R means "the real part of". It is an even function of θ and harmonic in the unit circle, and hence its Fourier series there is of the form

$$h_\beta(r, \theta) = 1 + \sum_{n=1}^{\infty} A_n(\beta)\, r^n \cos n\theta.$$

We assert that

$$\frac{m}{n^{\alpha}} \leq A_n(\beta) \leq \frac{M}{n^{\alpha}}$$

for every n.

Proof. (Where there can be no ambiguity, we shall sometimes write A_n for $A_n(\beta)$.) It is sufficient to show that

$$A_n > 0, \quad n = 1, 2, \ldots \tag{5.2}$$

and

$$\lim_{n=\infty} A_n \cdot n^{\alpha} = \text{const.} > 0. \tag{5.3}$$

Inequality (5.2) follows from

$$A_n = \binom{\beta + n - 1}{n} = \frac{\beta \cdot (\beta + 1) \cdot \cdots \cdot (\beta + n - 1)}{1 \cdot 2 \cdot \cdots \cdot n}.$$

Formula (5.3) follows from a known property of the Γ function: we give, however, a short proof:

$$\log A_n = \log \frac{\beta}{n} + \sum_{k=1}^{n-1} \log\left(1 + \frac{\beta}{k}\right) = \log \frac{\beta}{n} + \sum_{k=1}^{n-1} \left(\frac{\beta}{k} + O\left(\frac{1}{k^2}\right)\right) =$$

$$= \log \frac{1}{n} + \beta \sum_{k=1}^{n-1} \frac{1}{k} + C' + o(1)$$

where $C' = \log \beta + \sum_{k=1}^{\infty} O\left(\frac{1}{k^2}\right)$, from which

$$\log A_n = \log \frac{1}{n} + \beta \log n + \beta \cdot C + C' + o(1),$$

where C is EULERS's constant, or

$$\log A_n = \log \frac{1}{n^{1-\beta}} + \text{const.} + o(1)$$

and (5.3) follows.

24. — Lemma 7. *Let* $z = re^{i\theta}$ *be a complex number with* $|z| \leq 1$. *Then*

(5.4) $$m < \frac{h_\beta(r,\theta)}{\left|\frac{1}{1-z}\right|^\beta} \leq 1,$$

(where m has a new value).

Proof. Observe first that the function $u = \frac{1}{1-z}$ maps the unit circle $|z| \leq 1$ of the z-plane onto the halfplane $R(u) \geq \frac{1}{2}$ of the u-plane. Observe further that the principal branch of the function $w = u^\beta$ maps the halfplane $R(u) > 0$ on the sector $|\arg w| < \frac{\pi\beta}{2}$. Hence it follows that the principal branch of the function $w = \left(\frac{1}{1-z}\right)^\beta$ maps the unit circle $|z| \leq 1$ of the z-plane onto a part of the sector $|\arg w| < \frac{\pi\beta}{2}$ of the w-plane. And every point w of the mentioned sector satisfies the condition

$$\cos\frac{\pi\beta}{2} < \frac{R(w)}{|w|} \leq 1$$

from which (5.4) follows.

25. — Lemma 8. *Consider the function*

$$h_\gamma(\theta) = R\left(\frac{1}{1-e^{i\theta}}\right)^\gamma.$$

its Fourier series

$$h_\gamma(\theta) = 1 + \sum_{n=1}^{\infty} A_n(\gamma)\cos n\theta.$$

and the partial sums

$$h_{\gamma,n}(\theta) = 1 + \sum_{k=1}^{n} A_k(\gamma)\cos k\theta. \qquad (n = 1, 2, \ldots)$$

Here γ is defined in (5.1) *and A_n was used already in lemma 6. We assert that the functions*

$$1 - \frac{h_{\gamma,n}(\theta)}{h_\gamma(\theta)} \qquad (n = 1, 2, \ldots)$$

1° *are uniformly bounded in* $(0, 2\pi)$ *and* 2° *tend uniformly to zero in every interval* $(\delta, 2\pi - \delta)$, *where* $0 < \delta < \pi$.

In the proofs of this lemma and the following one we shall write briefly $h(\theta)$, $h_n(\theta)$, and later $h_\delta(\theta)$.

Proof. In view of (5. 4) it is sufficient to show these properties for the functions

$$f(z) = \frac{1}{(1-z)^\gamma} = 1 + \sum_{n=1}^{\infty} A_n z^n \qquad (z = e^{i\theta})$$

and

$$f_n(z) = 1 + \sum_{k=1}^{n} A_k z^k,$$

since

$$h(\theta) = Rf(e^{i\theta}) \quad \text{and} \quad h_n(\theta) = Rf_n(e^{i\theta}).$$

1° Abel's transformation gives

$$f_n(z) = (1 - A_1) + \sum_{k=1}^{n-1} S_k (A_k - A_{k+1}) + S_n A_n, \tag{5.5.}$$

where

$$S_k(z) = 1 + z + \cdots + z^k = \frac{1 - z^{k+1}}{1 - z}. \tag{5.6}$$

Furthermore by lemma 6

$$0 < A_k \leq \frac{M}{k^{1-\gamma}} \tag{5.7}$$

and

$$0 < A_k - A_{k+1} = A_k \left(1 - \frac{\gamma + k}{k+1}\right) = A_k \cdot \frac{1-\gamma}{k+1} < \frac{M}{k^{2-\gamma}}. \tag{5.8}$$

Give to z a fixed value $\neq 1$ and let n_0 be the largest of the numbers 1 and the integral part of $\frac{1}{|z-1|}$. Formulae (5. 5) and (5. 6) now yield

$$\frac{f_n(z)}{f(z)} = \frac{1 - A_1}{(1-z)^{-\gamma}} + \left[\sum_{k=1}^{n_0} + \sum_{k=n_0+1}^{n-1}\right] \frac{1 - z^{k+1}}{(1-z)^{1-\gamma}} (A_k - A_{k+1}) +$$

$$+ \frac{1 - z^{n+1}}{(1-z)^{1-\gamma}} A_n = P + Q + R + S.$$

Here we have by (5.7) and (5.8)

$$|P| < \frac{1}{|1-z|^{1-\gamma}} \leq 2^{\gamma},$$

$$|Q| < M \cdot |1-z|^{\gamma} \cdot \sum_{k=1}^{n_0} \frac{k+1}{k^{2-\gamma}} \leq 2M \cdot |1-z|^{\gamma} \cdot \sum_{k=1}^{n_0} \frac{1}{k^{1-\gamma}} <$$

$$< 2M \cdot |1-z|^{\gamma} \cdot \frac{n_0^{\gamma}}{\gamma} \leq \frac{M \cdot 2^{1+\gamma}}{\gamma},$$

$$|R| < \frac{2M}{|1-z|^{1-\gamma}} \sum_{k=n_0+1}^{\infty} \frac{1}{k^{2-\gamma}} < \frac{2M}{|1-z|^{1-\gamma}(1-\gamma)} \cdot \frac{1}{n_0^{1-\gamma}} <$$

$$< \frac{2^{2-\gamma} M}{|1-z|^{1-\gamma}(1-\gamma)} \cdot \frac{1}{(n_0+1)^{1-\gamma}} < \frac{2^{2-\gamma} M}{1-\gamma},$$

and

$$|S| \leq \frac{2}{|1-z|^{1-\gamma}} \cdot \frac{M}{n^{1-\gamma}} < 2M.$$

Hence

$$\left| \frac{f_n(z)}{f(z)} \right| \leq M'$$

where M' depends on γ but not on n or z, and item 1° is settled.

2° The above estimation of $|R|$ gives that, if $|z-1| \geq \delta > 0$ but z otherwise is arbitrary on the unit circle,

$$\left| 1 - \frac{f_n(z)}{f(z)} \right| = \left| \frac{\sum_{k=n+1}^{\infty} A_k z^k}{f(z)} \right| < \frac{4M}{\delta^{1-\gamma}(1-\gamma)} \cdot \frac{1}{n^{1-\gamma}},$$

and our proposition 2° follows.

26. — Lemma 9. *Consider the function $h_\gamma(\theta)$ of the previous lemma and put*

$$h_{\gamma,\delta}(\theta) = \frac{1}{\delta} \int_{\theta}^{\theta+\delta} h_\gamma(t)\, dt. \qquad (\delta \leq 0)$$

We assert that the function

$$1 - \frac{h_{\gamma,\delta}(\theta)}{h_\gamma(\theta)},$$

depending on the parameter δ, 1° *tends to zero as* $\delta \to 0$ *for every* $\theta \neq 2k\pi$, $k = 0, \pm 1, \ldots$, *and* 2° *is uniformly bounded.*

Proof. 1° By an elementary calculation one finds that $h(\theta)$ in the interval $(0, 2\pi)$ can be written

$$h(\theta) = \left(\frac{1}{2\sin\frac{\theta}{2}}\right)^{\gamma} \cdot \cos^{\gamma}\frac{(\pi-\theta)}{2}. \tag{5.9}$$

Hence $h(\theta)$ is continuous in a neighbourhood of every point $\theta \neq 2k\pi$, from which follows that $h_\delta(\theta) \to h(\theta)$ as $\delta \to 0$ at every such point.

2° Let x be a small *positive* number. By (5.9) we have that $h(x) = O(x^{-\gamma})$, from which

$$h_x(0) = \frac{1}{x}\int_0^x h(x)\,dx = \frac{1}{x}\int_0^x O(x^{-\gamma})\,dx = \frac{1}{x}O(x^{1-\gamma}) = O(x^{-\gamma}). \tag{5.10}$$

The function $h(\theta)$ is even and periodic, and hence it is sufficient to consider the interval $0 < \theta \leq \pi$. We suppose, to fix our arguments, that $|\delta| < \frac{\pi}{2}$.

For positive δ our proposition is evident; we always have that

$$\frac{h_\delta(\theta)}{h(\theta)} < \frac{h\left(\frac{3\pi}{2}\right)}{h(\pi)} = \text{const.}$$

Negative values of δ then remain. For $-\delta = \theta$ we have by (5.10) and (5.9)

$$\frac{h_\delta(\theta)}{h(\theta)} = \frac{h_\theta(0)}{h(\theta)} = \frac{O(\theta^{-\gamma})}{M\cdot\theta^{-\gamma}} < \text{const.} \tag{5.11}$$

For $0 < -\delta < \theta$ it is seen that $h_\delta(\theta) < h_\theta(0)$ (we have in fact by (5.9) that $h(\theta)$ decreases monotonically in the interval $(0, \pi)$), and (5.11) yields our assertion. If $\theta < -\delta \leq 2\theta$, then $h_\delta(\theta) < 2h_\theta(0)$ and we are brought back to the case (5.11). If finally $2\theta < -\delta$, then we have once more $h_\delta(\theta) < h_\theta(0)$. Hence lemma 9 is completely proved.

27. — Lemma 10. *Let e^{ix} and e^{iy} be two distinct points on the unit circle. Then*

$$(5.12)\quad K(x,y)=\int_0^{2\pi}\frac{1}{|e^{it}-e^{ix}|^{\gamma}}\cdot\frac{1}{|e^{it}-e^{iy}|^{\gamma}}\,dt\leq\frac{M}{|e^{ix}-e^{iy}|^{\beta}},$$

where β and γ are given in (5.1) *and M depends only on α.*

Proof. We suppose that $x<y\leq x+\pi$, which is no essential restriction. The integrand of (5.12) has the period 2π, and hence we can write

$$K(x,y)=\int_{x-\frac{\pi}{2}}^{x-\frac{3\pi}{2}}\frac{1}{|e^{it}-e^{ix}|^{\gamma}}\cdot\frac{1}{|e^{it}-e^{iy}|^{\gamma}}\,dt.$$

In the interval $\left(-\frac{3\pi}{2},\ \frac{3\pi}{2}\right)$ we have the inequality $|e^{it}-1|\geq$ $\geq\frac{\sqrt{8}}{3\pi}\cdot|t|$, from which

$$\frac{1}{|e^{it}-e^{ix}|}\leq\frac{3\pi}{\sqrt{8}}\cdot\frac{1}{|t-x|}\quad\text{and}\quad\frac{1}{|e^{it}-e^{iy}|}\leq\frac{3\pi}{\sqrt{8}}\cdot\frac{1}{|t-y|}$$

in the interval $\left(x-\frac{\pi}{2},\ x+\frac{3\pi}{2}\right)$. We then get

$$K(x,y)\leq M\cdot\int_{x-\frac{\pi}{2}}^{x+\frac{3\pi}{2}}\frac{dt}{|t-x|^{\gamma}\,|t-y|^{\gamma}}<M\int_{-\infty}^{\infty}\frac{dt}{|t-x|^{\gamma}\,|t-y|^{\gamma}},$$

M depending only on γ. Put $y-x=2a$, and introduce $t-\frac{x+y}{2}$ as a new variable; we then have

$$K(x,y)<M\cdot\int_{-\infty}^{\infty}\frac{dt}{|t+a|^{1-\frac{\alpha}{2}}\,|t-a|^{1-\frac{\alpha}{2}}},$$

and if $a \cdot t$ is now introduced as a new variable, it follows that

$$K(x, y) < M \frac{1}{a^{1-\alpha}} \int_{-\infty}^{\infty} \frac{dt}{|t+1|^{1-\frac{\alpha}{2}} |t-1|^{1-\frac{\alpha}{2}}} =$$

$$= M' \frac{1}{|x-y|^{1-\alpha}} < M' \frac{1}{|e^{ix} - e^{iy}|^{1-\alpha}},$$

since the last integral is convergent and its value depends only on α. Hence (5. 12) is proved.

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